Rectification
Lecture No. 1

Rectification

Definition:

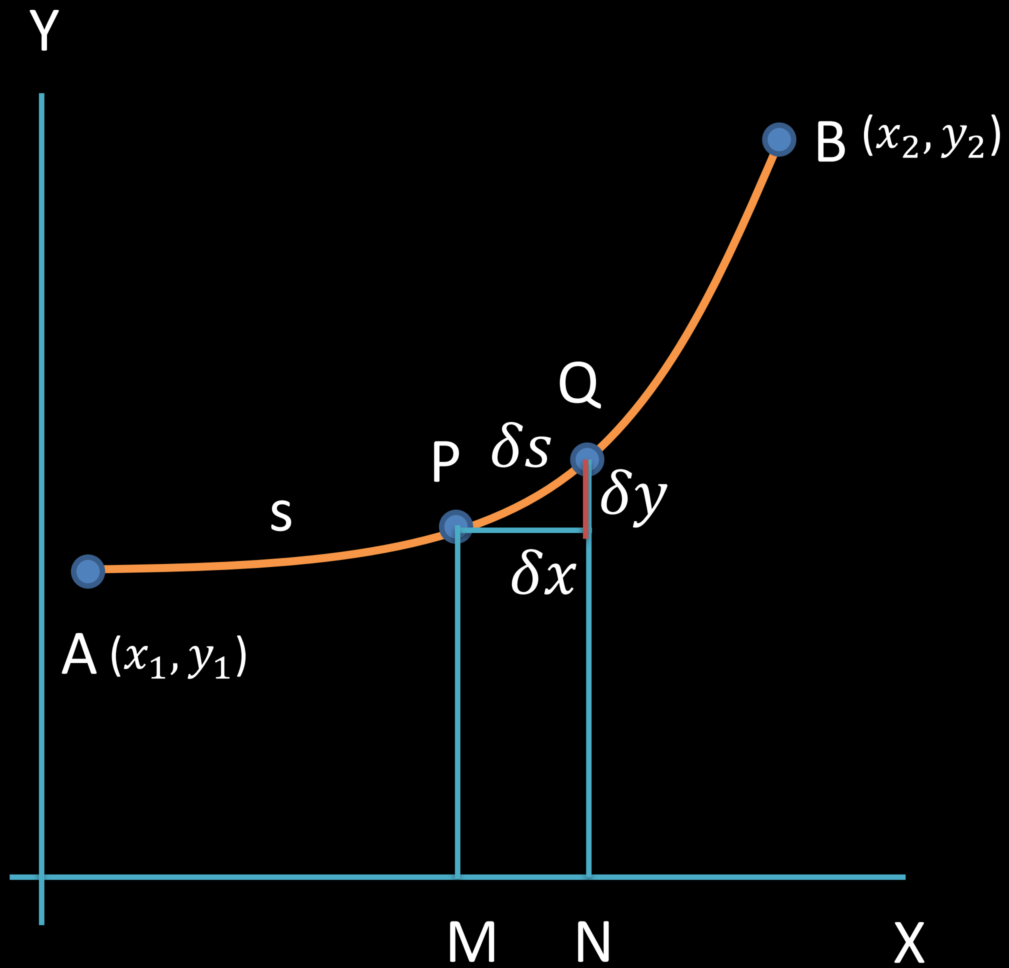
The technique of integration can be applied to find the lengths of given plane curves. This process is called as rectification

Types:

1. Using $\frac{dy}{dx}$
2. Using $\frac{dx}{dy}$
3. To find the length of a curve in parametric form
4. To find the length of a loop
5. Length of the arc of a curve given by $r = f(\theta)$

Derivation of formula

Let the equation of curve be y= f(x)



$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

$$\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

$$\left(\frac{\delta s}{\delta x}\right) = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

Taking the limits as $\delta x \to 0$

$$\left(\frac{ds}{dx}\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Types: Using $\frac{dy}{dx}$

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$$

Example 1:

Find the total length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$

Solution :

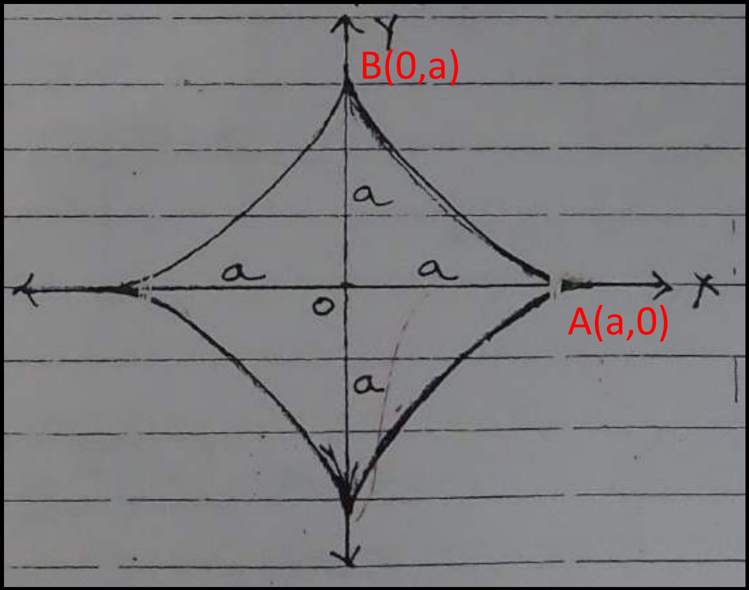
Differentiating the given equation

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$



$$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

$$s = \int_a^0 \sqrt{1 + \left(-\frac{y^{1/3}}{x^{1/3}}\right)^2}\ dx$$

$$s = \int_a^0 \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}}\ dx$$

$$s = \int_a^0 \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}}\ dx$$

$$s = \int_a^0 \sqrt{\frac{a^{2/3}}{x^{2/3}}}\ dx$$

$$s = \int_a^0 \frac{a^{1/3}}{x^{1/3}} dx$$

$$s = \int_a^0 a^{1/3}\ x^{-1/3}\ dx$$

$$s = a^{1/3} \int_a^0 x^{-1/3}\ dx$$

$$s = a^{1/3} \left[\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\right]_a^0$$

$$s = a^{1/3}\ \frac{3}{2}\ [0 - a^{2/3}]$$

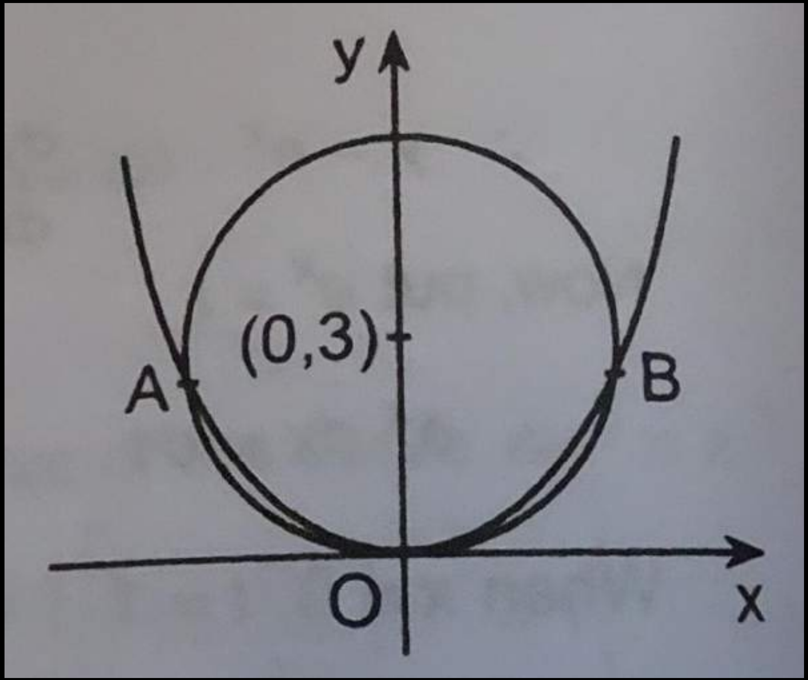
$$s = -\frac{3}{2}a$$

$\therefore$ *Total length of the curve*

$$s = 4\ X\ \frac{3}{2}a = 6a$$

Example 2:

Find the total length of the parabola $x^2 = 4y$ *which lies inside the circle* $x^2 + y^2 = 6y$



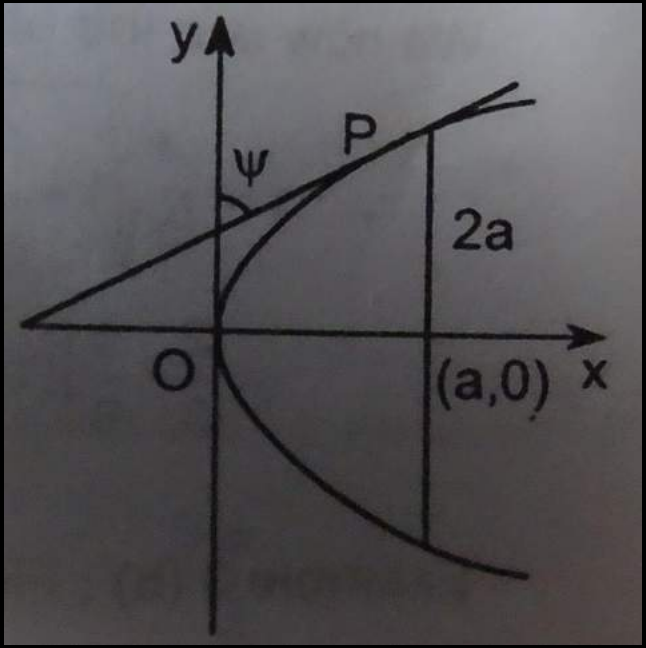
Type: Using $\frac{dx}{dy}$

$$s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

Example 1:

Show that the length of the parabola $y^2 = 4ax$ from the vertex to the end of the latus rectum is a$[\sqrt{2} + \log(1 + \sqrt{2})]$

Solution:



Let arc OP = s

$$x = \frac{y^2}{4a}$$

$$\frac{dx}{dy} = \frac{y}{2a}$$

$$\frac{dx}{dy} = \frac{y}{2a}$$

$$s = \int_0^{2a} \sqrt{1 + \left(\frac{y}{2a}\right)^2}\ dx$$

$$s = \int_0^{2a} \sqrt{1 + \frac{y^2}{4a^2}}\ dx$$

$$s = \frac{1}{2a}\int_0^{2a} \sqrt{4a^2 + y^2}\ dx$$

$$s = \frac{1}{2a}\left[\frac{y}{2}\sqrt{4a^2 + y^2} + \frac{4a^2}{2}\log(y + \sqrt{y^2 + 4a^2}\,)\right]$$

$$s = a\left[\sqrt{2} + \log(1 + \sqrt{2})\right]$$

Example 2:

Find the length of the curve $x = \frac{y^4}{4} + \frac{1}{8y^2}$ from y = 1 to y = 2

Thank You!
Any Question?